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Various entanglement measures do not give the same ordering for all quantum states in general [S. Virmani and M. B. Plenio, Phys. Lett. A, **268**, 31 (2000)]. That is, some two density operator's ordering in entanglement-degree depends on entanglement measures, which appears to be odd. However, we observe that any mixed states corresponding to such pair of density operators can not be transformed, with unit efficiency, to each other by any local operations. We discuss this fact in analogy with the relativity of temperal order in the special theory of relativity.

03.67.-a, 03.65.Bz

It is quantum entangled states that led to the controversy over Einstein-Podolsky-Rosen experiment [1] and later to Bell's inequality [2] that explicitly revealed nonlocal nature of quantum mechanics. On the other hand, entanglement is the key ingredient in quantum information processing: for example, the speedup in quantum computation [3] is obtained through the parallel quantum operations on massively superposed states which, we expect, are entangled in general.

For better understanding and manipulation of entangled states, we need to classify all quantum states as well as possible. Thus some authors have proposed a few entanglement measures that quantify entanglement of quantum states; Horodecki's proposed negative-eigenvalue-measure of entanglement E_N [4]- [6].[1] Bennett *et al.* invented three entanglement measures, that is, entanglement of formation E_F [2] and entanglement of distillation with one (two) way classical communications E_{D1} (E_{D2}) [7]. Vederal *et al.* proposed relative entropy of entanglement E_R [8]. Vederal and Plenio also considered an entanglement measure induced by Bures metric E_B and other numerous candidates [9]. All of these are *reasonable* ones in the sense that they satisfy the following three conditions [9].
(1) $E(\rho) = 0$ iff ρ is separable.
(2) Local unitary operations leave $E(\rho)$ invariant.
(3) The expected entanglement-degree cannot increase under any combination of local quantum operations (LQ), classical communication (CC), and post-selection (PS).
E_N, E_F, and E_R, E_B are shown to satisfy the conditions in Refs. [6], [7], and Ref. [9], respectively. It is easy to see that E_D's satisfy them from its definitions. All reasonable measures give rise to the same ordering for pure states, as we show later. So they can be re-scaled to become an identical one for pure states. However, the status of our knowledge about entanglement measures for mixed states is sometimes said to be still entangled; there is no unique one [10,9,7]. In this paper, we discuss the non-uniqueness (or relativity) of entanglement measures in analogy with the relativity of temporal order in the special theory of relativity [11]; temporal order of certain two events can be reversed depending on observer's reference frames. However, this fact does not give rise to contradictions because the two events cannot be time-likely connected. This is analogous to the following fact. Order of certain two mixed state's entanglement-degrees can be reversed depending on entanglement measures. However, this fact does not give rise to contradictions because the two mixed states cannot be transformed, with unit efficiency, to each other by any local operations, as we will see later. There might be no unique entanglement measure as if there is no preferred reference frame. Then we might as well search for methods that utilize multiplicity of reasonable entanglement measures rather than seek a unique one.

Certain two entanglement measures E_A and E_B are defined to have the same ordering if they satisfy the following condition for any density operator ρ_i and ρ_j.

$$E_A(\rho_i) > E_A(\rho_j) \Leftrightarrow E_B(\rho_i) > E_B(\rho_j) \qquad (0.1)$$

Let us show that all reasonable measures (that is, satisfying the three conditions) give rise to the same ordering for pure states: let us assume that there exist two pure states $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ that do not have the same ordering for certain two reasonable entanglement measures E_A and E_B. Then by definition we have either

$$E_A(\rho_1) > E_A(\rho_2) \text{ and } E_B(\rho_1) < E_B(\rho_2), \qquad (0.2)$$

or

$$E_A(\rho_1) < E_A(\rho_2) \text{ and } E_B(\rho_1) > E_B(\rho_2). \qquad (0.3)$$

[1] This measure can be defined in only two-dimensional bipartite systems to which we confine our discussion for simplicity.
[2] sometimes called as entanglement of creation

A quantum state with density operator ρ_i cannot be made, with unit efficiency, become one with ρ_j by any local operations, if entanglement-degree of ρ_i is less than ρ_j in any one of reasonable entanglement measures. Thus by Eqs. (0.2) and (0.3), ρ_1 cannot be transformed, with unit efficiency, into ρ_2 by any local operations and vice versa. However, this contradicts with the fact that pure state entanglement can be diluted with unit efficiency [12,7]. Q.E.D.

It is now easy to re-scale all entanglement measures to become an identical one for all pure states, since ordering of them are the same. In fact, the four entanglement measures E_F, E_D's, and E_R give the same value $h(\alpha^2) = -[\alpha^2 \log_2 \alpha^2 + (1-\alpha^2)\log_2(1-\alpha^2)]$ of entanglement-degree for a general pure entangled state $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$, where α and β are two real numbers satisfying $\alpha^2 + \beta^2 = 1$. However, for examples, negative-eigenvalue-measure and measure induced by Bures metric do not give the same value as those: $E_N(|\psi\rangle\langle\psi|) = 2\alpha\beta$ and $E_B(|\psi\rangle\langle\psi|) = (E_N)^2$ [9]. We can make both reduce to the unique one for pure states by the following transformations.

$$\tilde{E}_N = h[(1/2)(1+\sqrt{1-(E_N)^2})]$$
$$\text{and } \tilde{E}_B = h[(1/2)(1+\sqrt{1-E_B})]. \quad (0.4)$$

Recently Virmani and Plenio showed that if certain two entanglement measures, being identical for pure states, give different entanglement-degrees for some mixed states, ordering of the two measures must be different [10]: entanglement-degrees of pure states are continuously distributed between zero and one. Thus if ordering is given, entanglement-degrees of pure states determine those of mixed states. However, it has become clear that the various entanglement measures do not give the same entanglement-degree in general. For examples, existence of bound entangled state [13] and inevitable losses [9] in purification processes make E_F greater than E_D's. E_F and E_N have analytically closed forms [14,5] and thus it was explicitly shown that they give different entanglement-degree (and ordering in some cases) in general by computer calculation [5]. Therefore, we are led to a conclusion that various entanglement measures do not give the same ordering in general [10].

However, we note that although the conclusion appears to be odd, it does not give rise to bare contradictions: let us consider two mixed states with density operators ρ_a and ρ_b, respectively. The fact that ordering is reversed depending on entanglement measures implies that entanglement-degree of ρ_a is less than that of ρ_b in one of the two measures and vice versa. That is, an equation similar to Eqs. (0.2) and (0.3) is satisfied in this case, too. Thus, ρ_a cannot be transformed, with unit efficiency, into ρ_b by any local operations and vice versa. (It is true that with efficiency less than one the forbidden paths can be followed by local operations. However, it is worthwhile to know which paths are permitted by local operations with only unit efficiency.) So there is no contradiction. Now it is interesting to note that this fact is in analogy with the following one in the relativity theory. Temporal order of certain two events depends on observer's reference frames [11]. Although it appears to be odd, this fact does not give rise to contradictions because the two events cannot be time-likely connected. Then let us consider a map where each density operator is coordinated by entanglement-degrees of various measures, for example, E_F and $\tilde{E}_N$. (This is similar to Figs. 2 and 3 in Ref. [5].) We can easily see that a point in the map cannot be moved by any local unitary operations due to the condition (2). Thus a class of mixed states which are equivalent within local unitary operation corresponds to a point in the map. It does not seem that the converse is true in general. However, the larger number of reasonable entanglement measures we adopt, the more refined become the coordination of density operators. By applying general local quantum operations, a point can flow down through a trajectory that always points to lower-left direction from a point in the map, as if each observer goes through a path in space-time that always points to somewhere within light-cone from a point. Here a class of density operators corresponding to a point and the trajectory in the map are respectively in analogy with an event and an observer's path in space-time. This analogy might suggest that there is no unique entanglement measure as if there is no preferred reference frame. If so, we might as well make use of the multiplicity of entanglement measures as we did in coordinating density operators. Another possibility is that density operators have a multi-dimensional structure that are revealed as various facets depending on measures.

Here we showed that all reasonable entanglement measures give rise to the same ordering for pure states. Thus they can be made identical for pure states. Recently Virmani and Plenio have shown that if certain two entanglement measures which are identical for pure states, give different values of entanglement-degree for some mixed states, ordering of the two measures must be different. However, it has become clear that the various entanglement measures do not give the same value in general. Therefore some two density operator's ordering in entanglement-degree depends on entanglement measures, which appears to be odd. However, any mixed states corresponding to such pair of density operators can not be transformed to each other, with unit efficiency, by any local operations. We discussed this fact in analogy with the relativity of temporal order in the special theory of relativity.

ACKNOWLEDGMENTS

This work was supported by the Korean Ministry of Science and Technology through the Creative Research Initiatives Program under Contract No. 98-CR-01-01-A-20.

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[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935); F. Selleri, ed., *Quantum Mechanics versus Local Realism. The Einstein-Podolsky-Rosen Paradox* (Plenum, New York, 1988).

[2] J.S. Bell, Physics **1**, 195 (1964), reprinted in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press,1987).

[3] P. Shor, *Proc. 35th Ann. Symp. on Found. of Computer Science* (IEEE Comp. Soc. Press, Los Alomitos, CA, 1994) 124-134.

[4] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[5] J. Eisert and M. B. Plenio, J. Mod. Opt. **46**, 145 (1999).

[6] J. H. Lee, M.S. Kim, Y.J. Park, and S. Lee, J. Mod. Opt. **47**, 2151 (2000).

[7] C.H. Bennett, D.P. Divincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **54**(1996), 3824.

[8] V. Vederal, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[9] V. Vederal and M.B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[10] S. Virmani and M.B. Plenio, Phys. Lett. A **268**, 31 (2000).

[11] E.F. Taylor and J.A. Wheeler, *Spacetime Physics* (W.H. Freeman and Company, San Francisco and London, 1966).

[12] C.H. Bennett, H.J. Berstein, S. Popescu, B. Schmacher, Phys. Rev. A **53**, 2046 (1996).

[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **84**, 4260 (2000).

[14] W.K. Wootters. Phys. Rev. Lett. **80**, 2245 (1998).